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THE PROBLEM OF THE STABILITY OF QUASILINEAR FLOWS WITH RESPECT TO PERTURBATIONS OF THE HARDENING FUNCTION†

D. V. GEORGIEVSKII

Moscow (e-mail: georgiev@mech.math.msu.su; website: http://www.math.msu.su/~georgiev)

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A formulation of the linearized boundary-value problem of the stability of a deformation process with respect to small perturbations of the hardening function (of the scalar constitutive relation of the material) is presented. The characteristic vector relations of the medium are assumed to be linear. The occurrence of rigid zones in the domain of the solid and the change in their boundaries in the perturbed motion are taken into account. A perfect rigid plastic deformation and the flow of a Newtonian fluid are considered explicitly as the basic flow. In the latter case, the equation of the asymptotic boundary of the rigid zone, which appears when there is a small variation in the yield stress and a transition to a viscoplastic material, is derived.

The analysis of the stability of deformation processes with respect to a perturbation of the material functions is associated with the investigation of the solutions of differential equations when the coefficients vary [1]. Not only the constant coefficients are varied but also the functions and functionals which characterize the material, such as the hardening function, the creep and relaxation functionals and so on. Instability can be interpreted as a change of the type of boundary-value problem [2] which induces a qualitative change in the properties of the solution. Many classical paradoxes in mechanics and engineering applications (limit transitions, destabilization, spurious resonances, etc.) are also associated with this [3].

A formulation of the linearized problem of the stability of a deformation process for an incompressible material with linear vector relations (a problem of the first approximation) is presented below.

1. FORMULATION OF THE STABILITY PROBLEM

We will investigate the deformation of an incompressible solid with a scalar constitutive relation, taken in a fairly general form

$$T = T(U) \tag{1.1}$$

where $T(\mathbf{x}, t) = (s\colon s/2)^{1/2}$ is the maximum shear stress ($T = \sigma_u/\sqrt{2}$), and $U(\mathbf{x}, t) = (2\upsilon : \upsilon)^{1/2}$ is the maximum strain rate ($U = \sqrt{(2)}\upsilon_u$) [4]. The hardening function $T(U)$ of the material is continuous together with its first two derivatives with respect to its argument when $U > 0$. The vector (tensor) relations are assumed to be linear and, therefore, when account is taken of relation (1.1), the relation between the stress deviator, s, and the strain rate tensor υ has the form

$$s = 2T(U)v / U \tag{1.2}$$

The quantities appearing in relations (1.1) and (1.2) and all of the subsequent equations are reduced to dimensionless form in the basis $\{\rho; V; h\}$, where ρ is the constant density of the solids, and V and h are the characteristic velocity and linear dimension, respectively.

We will now write out, in orthogonal curvilinear coordinates, a formulation of the problem of a vectorially linear flow with constitutive relations (1.2) in a domain Ω of Eulerian space. The domain Ω consists of a flow zone Ω_f and a rigid zone Ω_r, the boundary Σ_r of which with Ω_f is determined at each instant of time from the equation

$$\Sigma_r = \{\mathbf{x}:\ T[U(\mathbf{x},t)] = \tau_s\} \tag{1.3}$$

where τ_s is the shear yield stress, which is one of the mechanical constants of the material.

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In the subdomain Ω_f, we have the incompressibility condition

$$\operatorname{tr} \upsilon = 0 \tag{1.4}$$

and the equations of motion with the mass forces **F**

$$-\operatorname{grad} p + 2\operatorname{Div}[T(U)\upsilon / U] + \mathbf{F} = d\mathbf{v}/dt \tag{1.5}$$

in which it is necessary to substitute the Stokes relations

$$\upsilon = \operatorname{Def} \mathbf{v} \tag{1.6}$$

in order to obtain a closed system of four equations in one vector function $\mathbf{v}(\mathbf{x}, t)$ and the single scalar function $p(\mathbf{x}, t)$.

The boundary conditions

$$\mathbf{x} \in \Sigma_v : \mathbf{v} = \mathbf{u}; \quad \mathbf{x} \in \Sigma_\sigma : -p\mathbf{n} + s \cdot \mathbf{n} = \mathbf{P} \tag{1.7}$$

are specified, generally speaking, on the moving surfaces $F_j(\mathbf{x}, t) \equiv 0$, the equations of which have the following form

$$d[F_j(\mathbf{x}, t)]/dt = 0, \quad \mathbf{x} \in \Sigma_j, \quad j = v, \sigma \tag{1.8}$$

If the motion of the medium is unsteady, it is also necessary to specify the initial conditions

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{w}(\mathbf{x}) \tag{1.9}$$

The solution of the initial-boundary-value problem of the vectorially linear flow (1.4)–(1.9) with the "initial data" (**F**; **u**; **P**; **w**) can only be found in the subdomain Ω_f of the body, which is not known beforehand. The surface Σ_r, which is defined by condition (1.3), is found when solving problem (1.4)–(1.9). Note that, if the latter surface is statically defineable (for example, for a quasistatic plane deformation of a perfect plastic solid or for pure shear of a viscoplastic solid) the stress distributions $\sigma(\mathbf{x}, t)$ and $T(\mathbf{x}, t)$ which are found hold both in the subdomain Ω_f as well as in the rigid zone Ω_r. In this case, the boundary Σ_r can immediately be found from Eq. (1.3), and the kinematics can then be found in the subdomain Ω_f which is now known.

We assume that, for a certain fixed scalar function $T^0(U)$, the solution of problem (1.4)–(1.9) with the "initial data" (**F**; **u**; **P**; **w**) is known and that the equation of the surface Σ_r^0 is also known. We call this solution, which is subsequently labelled with a zero superscript, the main or unperturbed solution.

Together with the main problem, we now consider problem (1.4)–(1.9) with the same "initial data" (**F**; **u**; **P**; **w**) but for a material with another scalar function $T(U)$ (the perturbed problem) where

$$T(U) = T^0(U) + \delta T(U) \equiv T^0(U) + \alpha T^1(U)$$
$$\delta T(U) \in C^{(1)}]0;+\infty[, \quad \alpha = \sup_{U>0} |\delta T(U)| \tag{1.10}$$

so that $|T^1(U)| \leqslant 1$.

Regarding α as a small parameter ($\alpha \ll 1$), we represent the solution of the perturbed problem in the following manner

$$p(\mathbf{x},t) = p^0(\mathbf{x},t) + \alpha p^1(\mathbf{x},t), \ldots, s(\mathbf{x},t) = s^0(\mathbf{x},t) + \alpha s^1(\mathbf{x},t) \tag{1.11}$$

and the equation of the surface Σ_r, which separates the rigid zone from the flow zone in the perturbed process, is determined from (1.3)

$$\Sigma_r = \{\mathbf{x}: \ T[U(\mathbf{x},t)] = \tau_s^0 + \alpha\tau_s^1\} \tag{1.12}$$

Note that

$$\tau_s^0 = \lim_{U\to 0} T^0(U), \quad \tau_s^1 = \lim_{U\to 0} T^1(U) \tag{1.13}$$

We substitute expressions (1.11) into (1.4)–(1.9) and take account of the fact that quantities with a zero superscript are solutions of the main problem. On equating the coefficients of α to zero in each of the equations, we obtain, in Ω_f, the following, as yet unclosed initial-boundary-value problem of the first approximation

$$\operatorname{tr} \upsilon^1 = 0 \tag{1.14}$$

$$-\operatorname{grad} p^1 + \operatorname{Div} s^1 = \frac{\partial \mathbf{v}^1}{\partial t} + (\mathbf{v}^0 \otimes \nabla)\cdot \mathbf{v}^1 + (\mathbf{v}^1 \otimes \nabla)\cdot \mathbf{v}^0 \tag{1.15}$$

$$\upsilon^1 = \operatorname{Def} \mathbf{v}^1 \tag{1.16}$$

$$\mathbf{x} \in \Sigma_\upsilon : \mathbf{v}^1 = 0; \quad \mathbf{x} \in \Sigma_\sigma : -p^1\mathbf{n} + s^1 \cdot \mathbf{n} = 0 \tag{1.17}$$

$$\mathbf{v}^1(\mathbf{x}, 0) = 0 \tag{1.18}$$

Next, substituting (1.11) into the non-linear constitutive relations (1.2), after some reduction we write the relation between the tensors s^1 and υ^1 or the analogue of the constitutive relations of the so-called medium of first approximation

$$s^1 = \frac{2}{U_0}\left[T^0(U^0)\upsilon^1 + T^1(U^0)\upsilon^0\right] + 2\left[T^{0\prime}(U^0) - \frac{T^0(U^0)}{U^0}\right]V^0 : \upsilon^1 \tag{1.19}$$

$$V^0 = \frac{2\upsilon^0 \otimes \upsilon^0}{U^{02}} \equiv \frac{\upsilon^0 \otimes \upsilon^0}{\upsilon^0 : \upsilon^0}$$

The fourth-rank tensor V^0 is determined by the kinematics of the main flow [5].

Here, account has been taken of the fact that, from the relations

$$U^0 + \alpha U^1 = U = \sqrt{2(\upsilon^0 + \alpha\upsilon^1) : (\upsilon^0 + \alpha\upsilon^1)} =$$

$$= \sqrt{2\upsilon^0 : \upsilon^0}\sqrt{1 + \frac{2\alpha\upsilon^0 : \upsilon^1}{\upsilon^0 : \upsilon^0} + O(\alpha^2)} = U^0\left(1 + \frac{2\alpha\upsilon^0 : \upsilon^1}{U^{02}}\right) + O(\alpha^2)$$

we have

$$U^1 = \frac{2\upsilon^0 : \upsilon^1}{U^0} \tag{1.20}$$

It follows from (1.20) that the quantity U^1 is not the maximum strain rate, constructed for the tensor υ^1, and can even take negative values.

Since υ^0 and U^0 are known from the basic motion, a medium of the first approximation is not only vectorially but also scalarly linear. This medium has initial stresses $2T^1(U^0)\upsilon^0/U^0$ which, after substituting (1.19) into (1.15) can be considered as fictitious mass forces. In the domain Ω_f, the system of equations in variables with the superscript one becomes closed. Note that second derivatives of T^0 with respect to U and first derivatives of T^1 with respect to U are introduced in (1.15). The continuity of these functions in Ω_f is previously stipulated by the classes of smoothness to which $T^0(U)$ and $T^1(U)$ belong.

Representation (1.11) with the subsequent investigation of the initial-boundary-value problem of the first approximation (1.4)–(1.9) and the determination of the asymptotic boundaries of the rigid zones from (1.12) is a perturbation of the known solution with respect to a material function (in this case the hardening function of the material). The closeness in one or another sense, when $t > 0$, of the perturbed and main solutions enables us to make a judgement regarding the stability or the sensitivity of the flow to variations of the material functions. Concepts of such a kind can be approached by generalizing the classical Lyapunov–Movchan method of stability with respect to two measures [6, 7].

We will now give a definition of the stability of the process of deformation with respect to initial perturbations and a variation of a scalar function [5].

Definition. An unperturbed deformation process is said to be stable with respect to pairs of measures $\{(\rho_1^\circ, \rho_1(t)), (\rho_2^\circ, \rho_2(t)), \ldots, (\rho_k^\circ, \rho_k(t))\}$ and with respect to measure γ if, for any set of positive numbers $\bar{\varepsilon} = \{\varepsilon_1, \ldots, \varepsilon_k\}$, numbers $\delta_1(\bar{\varepsilon}), \ldots, \delta_{k+1}(\bar{\varepsilon})$ exist such that, for any perturbed process, which satisfy the inequalities

$$\|\delta\mathbf{v}(\mathbf{x},0)\|_{\overset{\circ}{\rho}_1}<\delta_1,\quad \|\delta\sigma(\mathbf{x},0)\|_{\overset{\circ}{\rho}_2}<\delta_2,\ \ldots,\ \|\delta F_\alpha(\mathbf{x},0)\|_{\overset{\circ}{\rho}_k}<\delta_k,\quad \|\delta T(U(\mathbf{x},0))\|_\gamma<\delta_{k+1} \tag{1.21}$$

when $t = 0$, the inequalities

$$\|\delta\mathbf{v}(\mathbf{x},t)\|_{\rho_1}<\varepsilon_1,\quad \|\delta\sigma(\mathbf{x},t)\|_{\rho_2}<\varepsilon_2,\ldots,\ \|\delta F_\alpha(\mathbf{x},t)\|_{\rho_k}<\varepsilon_k \tag{1.22}$$

hold when $t > 0$.

The first k inequalities in (1.21) and all of the inequalities in (1.22) set limits on the initial and actual perturbations of all the kinematic and dynamic parameters of the motion (all k of them), including the equations of the body surfaces. The last inequality sets limits on the initial perturbation of the scalar function of the material. By virtue of (1.10), it is necessary to select the distance in the space of the functions from $C^{(1)}]0;+\infty[$ as a measure of the deviation γ of this perturbation.

We next present examples of problems of the first approximation for certain standard types of scalar functions (1.1) and perturbations (1.10).

2. THE STABILITY OF NEWTONIAN FLOWS WITH RESPECT TO PERTURBATIONS OF THE YIELD STRESS

In this case (Fig. 1)

$$T^0(U)=\frac{U}{\mathrm{Re}},\quad T(U)=\tau_s+\frac{U}{\mathrm{Re}},\quad \delta T(U)\equiv\tau_s$$
$$T^1(U)\equiv 1,\quad \tau_s^0=0,\quad \tau_s^1=1,\quad \alpha=\tau_s\ll 1 \tag{2.1}$$

where Re is the Reynolds number. Constitutive relations (1.19) of a medium of the first approximation take the form

$$s^1=2\left(\frac{v^0}{U^0}+\frac{v^1}{\mathrm{Re}}\right) \tag{2.2}$$

that is, the medium corresponds to a linear Newtonian model with initial stresses which are known from the main solution. In order to formulate the first approximation boundary-value problem, it is necessary to substitute relations (2.2) into (1.15) and (1.17).

After determining the tensor v^1 from this problem, we write Eq. (1.12) of the asymptotic boundary Σ_r, taking account of (2.1), as

$$\Sigma_r=\{\mathbf{x}:2\alpha v^0:v^1=-U^{02}\} \tag{2.3}$$

Problems of viscoplastic flows degenerating into viscous flows have been investigated in detail from the point of view of variational inequalities [8]. The following mathematical problem has been formulated as an unsolved problem: ". . . can it be proved (this is obvious from mechanical considerations) that domains, where the strain rates are equal to zero, become larger as the yield stress increases?" [8, Ch. VI, Section 5].

A partial answer to this question (for the case of low yield stresses) follows from Eqs (2.3). Actually, rigid zones, which do not exist in Newtonian flows, start to be formed around the points $\mathbf{x}\in\bar{\Omega}$ at which U^0 and T^0 are equal to zero. We will denote the set of such points at each instant of time by $\gamma^0(t)$: $\gamma^0(t)=\{\mathbf{x}\in\bar{\Omega}: U^0(\mathbf{x},t)=0\}$. Infinitely distant points of the body can also belong to the set $\gamma^0(t)$. Then, at any instant of time, the rigid zones are certain neighbourhoods of infinity.

The asymptotic enlargement of the domains Ω_r as τ_s increases is also of interest in practical problems involving viscoplastic flow where an exact solution for any τ_s is difficult. At the same time, the solutions of the corresponding viscous problems are well known. Such examples include the Hamel problem of the radial flow of a viscous fluid in a planar convergent channel and the Karman problem (in this problem, an "inflow" of the medium from infinity accompanying a perturbation of the yield stress, which is well known in the Newtonian case, will obviously not occur since, starting from a certain height, the axis of rotation will belong to the rigid zone [9]).

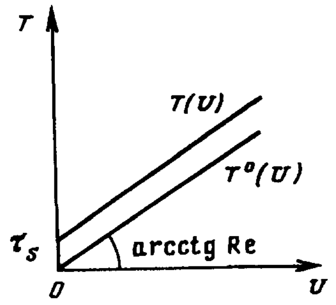

Fig. 1.

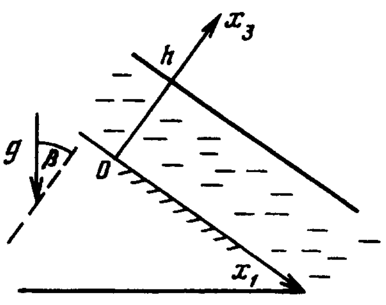

Fig. 2.

Below, we present an example of a purely illustrative nature where the plane-parallel shear of a heavy viscous layer on an inclined plane (Fig. 2) is taken as the main flow. In the Cartesian coordinates (Ox_1x_3), the known steady-state solution of the unperturbed problem has the form

$$v_1^0(x_3)=\frac{\text{Re}}{2}x_3(2-x_3)\sin\beta,\ \ v_3^0\equiv 0,\ \ v_{\alpha\alpha}^0\equiv 0$$

$$v_{13}^0(x_3)=\frac{U^0}{2}=\frac{\text{Re}}{2}(1-x_3)\sin\beta$$

$$p_0(x_3)=(1-x_3)\cos\beta,\ \ s_{\alpha\alpha}^0\equiv 0,\ \ s_{13}^0(x_3)=T^0=(1-x_3)\sin\beta$$

The characteristic velocity V is taken to be equal to $\sqrt{(gh)}$.

The boundary-value problem of the first approximation is formulated as follows:

$$v_{i,i}^1=0$$

$$-p_i^1+s_{ij,j}^1=\frac{\partial v_i^1}{\partial t}+\text{Re}\,x_3(1-x_3)v_3^1\delta_{1i}\sin\beta+\frac{\text{Re}}{2}x_3(2-x_3)v_{i,1}^1\sin\beta$$

$$s_{\alpha\alpha}^1=\frac{2}{\text{Re}}v_{\alpha\alpha}^1,\ \ s_{13}^1=1+\frac{1}{\text{Re}}(v_{1,3}^1+v_{3,1}^1)$$

$$x_3=0:\ v_i^1=0,\ \ x_3=1:\ p^1=s_{13}^1=0$$

Finding its solution in Ω in the class of plane-parallel fields ($v_1^1=v_1^1(x_3)$, $v_3^1\equiv 0$), we obtain

$$v_1^1(x_3)=-\text{Re}\,x_3,\ \ v_{\alpha\alpha}^1\equiv 0,\ \ v_{13}^1\equiv-\frac{\text{Re}}{2},\ \ p^1\equiv 0,\ \ s_{ij}^1\equiv 0$$

The set $\gamma^0(t)$ consists of the straight line $x_3 = 1$, and the asymptotic boundary of the rigid zone is determined from (2.3)

$$x_3 = 1-\alpha/\sin\beta \tag{2.4}$$

Hence, a rigid crust of thickness $\alpha/\sin\beta$ is formed close to the free boundary of the layer when there is a small perturbation of the yield stress. Note that the solution of the problem of viscoplastic flow along an inclined plane in a gravitational field, which is obtained after substitution into (1.11), as well as formula (2.4), are correct not just in the case of small α but for any finite α. So, if $\alpha \geqslant \sin\beta$, the shearing forces are insufficient for deformation and the whole layer is at rest.

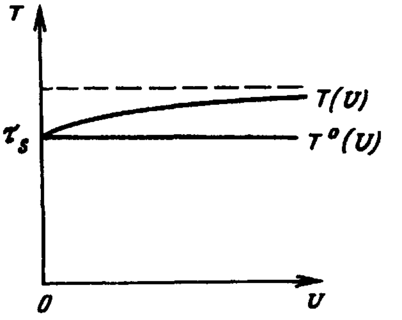

Fig. 3.

3. THE STABILITY OF PERFECT RIGID PLASTIC FLOWS (SAINT-VENANT FLOWS) WITH RESPECT TO PERTURBATIONS OF THE VISCOSITY

Suppose (Fig. 3) that

$$T^0(U) \equiv \tau_s, \quad T^{1'}(U) > 0, \quad \tau_s^1 = 0, \quad U > 0 \tag{3.1}$$

which corresponds to the addition of a small non-linear viscosity (hardening) in a perfect rigid plastic Saint-Venant model. From (1.9), we have

$$s^1 = \frac{2T^1(U^0)}{U^0} v^0 + \frac{2\tau_s}{U^0}(\Delta - V^0) : v^1 \tag{3.2}$$

where Δ is a unit fourth-rank tensor. By solving the linearized initial-boundary-value problem of the first approximation (1.14)–(1.18), (3.2), it is possible to find the kinematics **v** and the tensor υ^1. However, the components of this tensor will not participate in expression (1.12) for the asymptotic boundary Σ_r. In fact, substituting relations (3.1) into Eq. (1.12), we have

$$\Sigma_r = \{\mathbf{x}: \; T^1(U^0) = 0\} = \{\mathbf{x}: \; U^0 = 0\} \tag{3.3}$$

that is, on passing from the unperturbed motion (Saint-Venant flow) to the perturbed equation of the boundaries of the rigid zones, they remain the same.

As in Section 2, the problem of the first approximation is of interest in well-known classical solutions for a perfect rigid plastic medium. The solutions of many of the quasistatic problems which have been collected together in [10] are included among such classical solutions. These flows simulate the processes involved in the treatment of material with pressure, motions over surfaces, in thin films, etc. [11].

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